Existence and Multiplicity of Solutions for Nonlocal Neumann Problem with Non-Standard Growth

Francisco Julio S.A. Corrêa *
Universidade Federal de Campina Grande
Centro de Ciências e Tecnologia
Unidade Acadêmica de Matemática
CEP:58.109-970, Campina Grande - PB - Brazil
E-mail: fjsacorrea@gmail.com

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Augusto César dos Reis Costa
Universidade Federal do Pará
Instituto de Ciências Exatas e Naturais
Faculdade de Matemática
CEP:66.075-110, Belém - PA - Brazil
E-mail: aug@ufpa.br

June 22, 2015

Abstract

In this paper we are concerning with questions of existence of solution for a nonlocal and non-homogeneous Neumann boundary value problems involving the $p(x)$-Laplacian in which the non-linear terms have critical growth. The main tools we will use are the generalized Sobolev spaces and the Mountain Pass Theorem.

MSC: 35J60; 35J70; 58E05
Keywords: nonlocal problem; Neumann boundary conditions; trace; Sobolev spaces with variable exponent; critical exponent.

*Partially supported by CNPq-Brazil under Grant 301807/2013-2.
1 Introduction

In this paper we are going to study questions of existence of solutions for the nonlocal and non-homogeneous equation, with critical growth and Neumann boundary conditions, given by

$$M \left( \int \Omega \left| \nabla u \right|^{p(x)} + |u|^{p(x)} \right) \left( -\text{div}(|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u \right)$$

$$= \lambda f(x, u) \left[ \int \Omega F(x, u)dx \right]^r \quad \text{in } \Omega,$$

$$M \left( \int \Omega \left| \nabla u \right|^{p(x)} + |u|^{p(x)} \right) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu}$$

$$= \gamma g(x, u) \left[ \int_{\partial \Omega} G(x, u)ds \right]^{\kappa} \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of $\mathbb{R}^N$, $p \in C(\Omega)$, $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \partial \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous functions enjoying some conditions which will be stated later, $F(x, u) = \int_0^u f(x, \xi) d\xi$, $G(x, u) = \int_0^u g(x, y) dy$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $\lambda, r, \gamma, \kappa$ are real parameters, and $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator, that is,

$$\Delta_{p(x)} u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), \quad 1 < p(x) < N.$$

Bonder and Silva [3], in 2010, extended the concentration-compactness principle by Lions [16] for variable exponent spaces and applied the result to the Dirichlet problem involving the $p(x)$-Laplacian with subcritical and critical growth and showed the existence and multiplicity of solutions via Mountain Pass Theorem, Truncation argument and Krasnoselskii genus, inspired in Azorero and Peral [2], who solved the problem for the $p$-Laplacian. It is important to note that there exists similar result due to Bonder and Silva [3], on concentration-compactness principle, obtained by Fu [13]. Liang and Zhang [15], in 2011, showed multiplicity of solutions to the Neumann problem involving the $p(x)$-Laplacian, with critical growth, via truncation argument. Guo and Zhao [14], in 2012, showed existence and multiplicity of solutions for the nonlocal Neumann problem involving the $p(x)$-Laplacian, with subcritical growth, using Mountain Pass Theorem and Fountain Theorem. Bonder, Saintier and Silva [4], in 2013, presented the concentration-compactness principle for variable exponent in the trace case and in [5] showed existence of solution for the Neumann problem with critical growth on the boundary of the domain, via Mountain Pass Theorem.

In this article, we discuss existence of solutions for the nonlocal problem (1.1) with critical growth on the domain and on its boundary. We study the nonlocal condition
for the two following important classes of functions: \( M(\tau) = a + br^\eta \) with \( a \geq 0, b > 0, \eta \geq 1 \) and \( m_0 \leq M(\tau) \leq m_1 \), where \( m_0 \) and \( m_1 \) are positive constants. Note that the original Kirchhoff term is included in our analysis.

We will study the problem with the following critical Sobolev exponents

\[
p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{and} \quad p_*(x) = \frac{(N - 1)p(x)}{N - p(x)}, \tag{1.2}
\]

where \( p_* \) is critical exponent from the point of view of the trace.

Problems in the form (1.1) are associated with the energy functional

\[
J_{\lambda,\gamma}(u) = \bar{M} \left( \int_{\Omega} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \right) - \frac{\lambda}{r + 1} \left[ \int_{\Omega} F(x, u) \, dx \right]^{r+1} - \frac{\gamma}{\kappa + 1} \left[ \int_{\partial \Omega} G(x, u) \, ds \right]^{\kappa+1} \tag{1.3}
\]

for all \( u \in W^{1,p(x)}(\Omega) \), where \( \bar{M}(t) = \int_{0}^{t} M(s) \, ds \), \( ds \) denotes the boundary measure, and \( W^{1,p(x)}(\Omega) \) is the generalized Lebesgue-Sobolev space whose precise definition and properties will be established in section 2.

Depending on the behavior of the functions \( p, q \) the above functional is differentiable and its Fréchet-derivative is given by

\[
J'_{\lambda,\gamma}(u)v = M \left( \int_{\Omega} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \right) \int_{\Omega} \left( |\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2}uv \right) \, dx - \lambda \left[ \int_{\Omega} F(x, u) \, dx \right]^r \int_{\Omega} f(x, u)v \, dx - \gamma \left[ \int_{\partial \Omega} G(x, u) \, ds \right]^{\kappa} \int_{\partial \Omega} g(x, u)v \, ds \tag{1.4}
\]

for all \( u, v \in W^{1,p(x)}(\Omega) \). So, \( u \in W^{1,p(x)}(\Omega) \) is a weak solution of problem (1.1) if, and only if, \( u \) is a critical point of \( J_{\lambda,\gamma} \).

**Theorem 1.1**

(i) Assume \( \kappa = 0, g(x, u) = |u|^{q(x)-2}u, q : \partial \Omega \to [1, \infty) \) and \( A := \{ x \in \partial \Omega : q(x) = p_*(x) \} \neq \emptyset \) and \( M(\tau) = a + br^\eta \), with \( a \geq 0, b > 0, \eta \geq 1 \). Moreover, assume the existence of functions \( p(x), q(x), \beta(x) \in C_+^{+}(\Omega) \), see section 2, positive constants \( A_1, A_2 \) such that \( A_1 t^{\beta(x)-1} \leq f(x, t) \leq A_2 t^{\beta(x)-1} \) for all \( t \geq 0 \) and for all \( x \in \Omega \), with \( f(x, t) = 0 \) for all \( t < 0 \). Furthermore, \( (\eta + 1)p^+ < \beta^-(r+1) < \eta^+ \) and \( (\eta + 1)(p^+)^{\eta+1} < \frac{(\beta^+)^{\eta+1}(r+1)}{(\beta^-)^r} \). Then there exists \( \lambda_0 > 0 \) such that for all \( \lambda > \lambda_0 \) and for all \( \gamma > 0 \) there exists a nontrivial solution to (1.1).
section 2, positive constants

(ii) Assume \( \kappa = 0, g(x,u) = |u|^{q(x)-2}u, q : \partial \Omega \to [1,\infty) \) and \( A := \{ x \in \partial \Omega : q(x) = p_*(x) \} \neq \emptyset \). Moreover, assume the existence of functions \( p(x), q(x), \beta(x) \in C_+(\overline{\Omega}) \), see section 2, positive constants \( A_1, A_2 \) such that \( A_1 t^{\beta(x)-1} \leq f(x,t) \leq A_2 t^{\beta(x)-1} \) for all \( t \geq 0 \) and for all \( x \in \Omega \), with \( f(x,t) = 0 \) for all \( t < 0 \). Furthermore, assume there exists \( 0 < m_0 \) and \( m_1 \) such that \( m_0 \leq M(\tau) \leq m_1 \), with \( \frac{m_1 p^+}{m_0} < \left( \frac{A_1}{A_2} \right)^{r+1} \frac{(\beta^-)^{r+1}(r+1)}{(\beta^+)^r} \) and \( p^+ < \beta^-(r+1) < q^- \). Then there exists \( \lambda_1 > 0 \) such that for all \( \lambda > \lambda_1 \) and for all \( \gamma > 0 \) there exists a nontrivial solution to (1.1).

(iii) Suppose \( r = 0, f(x,u) = |u|^{q(x)-2}u, q : \Omega \to [1,\infty) \) and \( A := \{ x \in \Omega : q(x) = p^*(x) \} \neq \emptyset \). Moreover, assume the existence of functions \( p(x), q(x), \beta(x) \in C_+(\partial \Omega) \), positive constants \( A_1, A_2 \) such that \( A_1 t^{\beta(x)-1} \leq g(x,t) \leq A_2 t^{\beta(x)-1} \) for all \( t \geq 0 \) and for all \( x \in \partial \Omega \), with \( g(x,t) = 0 \) for all \( t < 0 \). Furthermore, assume there exists \( 0 < m_0 \) and \( m_1 \) such that \( m_0 \leq M(\tau) \leq m_1 \), with \( \frac{m_1 p^+}{m_0} < \left( \frac{A_1}{A_2} \right)^{\kappa+1} \frac{(\beta^-)^{\kappa+1}(\kappa+1)}{(\beta^+)^\kappa} \) and \( p^+ < \beta^-(\kappa+1) < q^- \). Then there exists \( \gamma_1 > 0 \) such that for all \( \gamma > \gamma_1 \) and for all \( \lambda > 0 \) there exists a nontrivial solution to (1.1).

(iv) Suppose \( r = 0, f(x,u) = |u|^{q(x)-2}u, q : \Omega \to [1,\infty) \) and \( A := \{ x \in \Omega : q(x) = p^*(x) \} \neq \emptyset \) and \( M(\tau) = a + b r^\eta, \) com \( a \geq 0, b > 0, \tau \geq 0, \) and \( \eta \geq 1 \). Moreover, assume the existence of functions \( p(x), q(x), \beta(x) \in C_+(\partial \Omega) \), positive constants \( A_1, A_2 \) such that \( A_1 t^{\beta(x)-1} \leq g(x,t) \leq A_2 t^{\beta(x)-1} \) for all \( t \geq 0 \) and for all \( x \in \partial \Omega \), with \( g(x,t) = 0 \) for all \( t < 0 \). Furthermore, assume \( (\eta + 1)p^+ < \beta^- (\kappa+1) < q^- \) and \( \frac{(\eta + 1)(p^+)^{\eta+1}}{(p^-)^{\eta}} < \left( \frac{A_1}{A_2} \right)^{\kappa+1} \frac{(\beta^-)^{\kappa+1}(\kappa+1)}{(\beta^+)^\kappa} \). Then there exists \( \gamma_2 > 0 \) such that for all \( \gamma > \gamma_2 \) and for all \( \lambda > 0 \) there exists a nontrivial solution to (1.1).

We should point out that the novelty in the present paper is the appearance of the integral terms, \( \left[ \int_{\Omega} F(x,u)dx \right]^\kappa \) and \( \left[ \int_{\partial \Omega} G(x,u)dS \right]^\kappa \), the critical growth on Neumann boundary conditions, the use of the \( p(x) \)-Laplacian and the generalized Lebesgue-Sobolev spaces.

We should point out that some ideas contained in this paper were inspired by the articles Corrêa and Figueiredo [[6], [7]], Fan [[8], [9]], Fu [13], Liang and Zhang [15], Mihailescu and Radulescu [17] and Yao [20].

This paper is organized as follows: In section 2 we present some preliminaries on the variable exponent spaces. In section 3, we proof of our main result.

2 Preliminaries on variable exponent spaces

First of all, we set

\[ C_+(\overline{\Omega}) := \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega} \} \]
and for each $h \in C_+(\bar{\Omega})$ we define 

$$h^+ := \max_{\bar{\Omega}} h(x) \quad \text{and} \quad h^- := \min_{\bar{\Omega}} h(x).$$

We denote by $\mathcal{M}(\Omega)$ the set of real measurable functions defined on $\Omega$. For each $p \in C_+(\bar{\Omega})$, we define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \left\{ u \in \mathcal{M}(\Omega); \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$ 

We consider $L^{p(x)}(\Omega)$ equipped with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$ 

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad \text{for all} \quad x \in \bar{\Omega}.$$ 

The proof of the following propositions may be found in Bonder and Silva [3], Bonder, Saintier and Silva [[4], [5]], Fan, Shen and Zhao [10], Fan and Zhang [11] and Fan and Zhao [12].

**Proposition 2.1** (Hölder Inequality) If $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, then

$$\left| \int_{\Omega} uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)}|v|_{p'(x)}.$$ 

**Proposition 2.2** Set $\rho(u) := \int_{\Omega} |u(x)|^{p(x)} dx$. For all $u \in L^{p(x)}(\Omega)$, we have:

1. For $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;
2. $|u|_{p(x)} < 1 \quad (= 1; > 1) \Leftrightarrow \rho(u) < 1 \quad (= 1; > 1)$;
3. If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
4. If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
5. $\lim_{k \to +\infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k \to +\infty} \rho(u_k) = 0$;
6. $\lim_{k \to +\infty} |u_k|_{p(x)} = +\infty \Leftrightarrow \lim_{k \to +\infty} \rho(u_k) = +\infty$.  

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The generalized Lebesgue - Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^p(\Omega); |\nabla u| \in L^p(\Omega) \right\}$$

with the norm

$$\| u \| := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Denoting $\rho_{1,p(x)} := \int_{\Omega}(|u|^{p(x)} + |\nabla u|^{p(x)})dx \quad \forall u \in W^{1,p(x)}(\Omega)$, we have the following proposition:

**Proposition 2.3** For all $u, u_j \in W^{1,p(x)}(\Omega)$, we have:

1. $\| u \| < 1 \ (= 1; > 1) \iff \rho_{1,p(x)}(u) < 1 \ (= 1; > 1)$;
2. If $\| u \| > 1$, then $\| u \|^{p^-} \leq \rho_{1,p(x)}(u) \leq \| u \|^{p^+}$;
3. If $\| u \| < 1$, then $\| u \|^{p^+} \leq \rho_{1,p(x)}(u) \leq \| u \|^{p^-}$;
4. $\lim_{j \to +\infty} \| u_j \| = 0 \iff \lim_{j \to +\infty} \rho_{1,p(x)}(u_j) = 0$;
5. $\lim_{j \to +\infty} \| u_j \| = +\infty \iff \lim_{j \to +\infty} \rho_{1,p(x)}(u_j) = +\infty$.

**Proposition 2.4** Suppose that $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ and $p \in C(\Omega)$ with $p(x) < N$ for all $x \in \Omega$. If $p_1 \in C(\Omega)$ and $1 \leq p_1(x) \leq p^*(x)$ ($1 \leq p_1(x) < p^*(x)$) for $x \in \Omega$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N - p(x)}$.

The Lebesgue spaces on $\partial \Omega$ are defined as

$$L^{q(x)}(\partial \Omega) := \left\{ u | u : \partial \Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial \Omega} |u|^{q(x)}dS < \infty \right\},$$

and the corresponding (Luxemburg) norm is given by

$$\| u \|_{L^{q(x)}(\partial \Omega)} := \| u \|_{q(x),\partial \Omega} := \inf\{\lambda > 0 : \int_{\partial \Omega} \left| \frac{u(x)}{\lambda} \right|^{q(x)}dS \leq 1\}.$$

**Proposition 2.5** Suppose that $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ and $p, q \in C(\Omega)$ with $p(x) < N$ for all $x \in \Omega$. Then there is a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial \Omega)$, where $q(x) < p_*(x) = \frac{(N-1)p(x)}{N - p(x)}$. 
Proposition 2.6 (Fan and Zhang [11]) Let $L_{p(x)} : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))'$ be such that

$$L_{p(x)}(u)(v) = \int_{\Omega} |\nabla u|^{p(x)} - 2\nabla u \nabla v dx, \ \forall \ u, v \in W^{1,p(x)}(\Omega),$$

then

(i) $L_{p(x)} : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))'$ is a continuous, bounded and strictly monotone operator;

(ii) $L_{p(x)}$ is a mapping of type $S_+$, i.e., if $u_n \to u$ in $W^{1,p(x)}(\Omega)$ and $\limsup(L_{p(x)}(u_n) - L_{p(x)}(u), u_n - u) \leq 0$, then $u_n \to u$ in $W^{1,p(x)}(\Omega)$;

(iii) $L_{p(x)} : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))'$ is a homeomorphism.

Proposition 2.7 (Bonder Saintier and Silva [4]) Let $(u_j)_{j \in \mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a sequence such that $u_j \to u$ weakly in $W^{1,p(x)}(\Omega)$. Then there exists a countable set $I$ positive numbers $(\mu)_i \in I$ and $(\nu)_i \in I$ and points $\{x_i\} \subset A := \{x \in \partial \Omega : q(x) = p_*(x)\}$ such that

$$|u_j|^{q(x)} dS \to \nu = |u|^{q(x)} dS + \sum_{i \in I} \nu_i \delta_{x_i}, \ \text{weakly} - \ast \text{ in the sense of measures.} \ (2.5)$$

$$|\nabla u_j|^{p(x)} dx \to \mu \geq |\nabla u|^{p(x)} dx + \sum_{i \in I} \mu_i \delta_{x_i}, \ \text{weakly} - \ast \text{ in the sense of measures.} \ (2.6)$$

$$\mathcal{T}_{x_i} \mu_i^{\frac{1}{p(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}, \ (2.7)$$

where $\mathcal{T}_{x_i} = \sup_{e > 0} T(p(\cdot), q(\cdot), \Omega_{\epsilon,i}, \Gamma_{\epsilon,i})$ is the localized Sobolev trace constant where

$$\Omega_{\epsilon,i} = \Omega \cap B_\epsilon(x_i) \ \text{and} \ \Gamma_{\epsilon,i} = \partial B_\epsilon(x_i) \cap \Omega.$$

Proposition 2.8 (Bonder and Silva [3]) Let $q(x)$ and $p(x)$ be two continuous functions such that

$$1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < N \ \text{and} \ 1 \leq q(x) \leq p_*(x) \ \text{in} \ \Omega.$$

Let $(u_j)$ be a weakly convergent sequence in $W^{1,p(x)}(\Omega)$ with weak limit $u$ and such that:

$$|\nabla u_j|^{p(x)} \to \mu \ \text{weakly} - \ast \text{ in the sense of measures.}$$

$$|u_j|^{q(x)} \to \nu \ \text{weakly} - \ast \text{ in the sense of measures.}$$

In addition we assume that $A := \{x \in \Omega : q(x) = p_*(x)\}$ is nonempty. Then, for some countable index set $I$, we have:

$$\nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, \ \nu_i > 0$$
\[
\mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_i, \quad \mu_i > 0
\]

\[
S_q \mu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)}, \quad \forall i \in I.
\]

Where \(\{x_i\}_{i \in I} \subset \mathcal{A}\) and \(S_q\) is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

\[
S_q = S_q(\Omega) = \inf_{\phi \in C_0^\infty(\Omega)} \frac{\||\nabla \phi|\|_{L^{p(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}.
\]

**Definition 2.1** We say that a sequence \((u_j) \subset W^{1,p(x)}(\Omega)\) is a Palais-Smale sequence for the functional \(J_{\lambda,\gamma}\) if

\[
J_{\lambda,\gamma}(u_j) \to c_* \quad \text{and} \quad J'_{\lambda,\gamma}(u_j) \to 0 \quad \text{in} \quad (W^{1,p(x)}(\Omega))^\prime,
\]

where

\[
c_* = \inf_{h \in C} \sup_{t \in [0,1]} J_{\lambda,\gamma}(h(t)) > 0
\]

and

\[
C = \{h : [0,1] \to W^{1,p(x)}(\Omega) : h \text{ continuous and } h(0) = 0, J_{\lambda,\gamma}(h(1)) < 0\}.
\]

The number \(c_*\) is called the energy level \(c_*\).

When (2.8) implies the existence of a subsequence of \((u_j)\), still denoted by \((u_j)\), which converges in \(W^{1,p(x)}(\Omega)\), we say that \(J_{\lambda,\gamma}\) satisfies the Palais-Smale condition.

### 3 Proof of Theorem 1.1

**Proof.** (i) The proof follows from several lemmas.

**Lemma 3.1** If \((u_j) \subset W^{1,p(x)}(\Omega)\) is a Palais-Smale sequence, with energy level \(c\), then \((u_j)\) is bounded in \(W^{1,p(x)}(\Omega)\).

**Proof.** Since \((u_j)\) is a Palais-Smale sequence with energy level \(c\), we have \(J_{\lambda,\gamma}(u_j) \to c\) and \(J'_{\lambda,\gamma}(u_j) \to 0\). Then taking \(\theta\) such that

\[
\frac{(\eta + 1)(p^+)^{r+1}}{(p^-)^{\eta}} < \theta < \frac{(\beta^-)^{r+1}(r + 1)}{(\beta^+)^{r}}
\]

\[
C + \|u_j\| \geq \left( J_{\lambda,\gamma}(u_j) - \frac{1}{\theta} J'_{\lambda,\gamma}(u_j) u_j \right) \geq
\]

\[
a \left( \int_{\Omega} \frac{1}{p(x)} \left( |\nabla u_j|^{p(x)} + |u_j|^{p(x)} \right) dx \right) + \frac{b}{\eta + 1} \left( \int_{\Omega} \frac{1}{p(x)} \left( |\nabla u_j|^{p(x)} + |u_j|^{p(x)} \right) dx \right)^{\eta+1}
\]

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where necessary, we get
\[\lambda \left( \int_{\Omega} F(x, u_j) dx \right)^{r+1} - \gamma \int_{\partial \Omega} \frac{1}{q(x)} |u_j|^{q(x)} dS - \frac{a}{\theta} \int_{\Omega} \nabla u_j \nabla u_j dx \]
\[- \frac{a}{\theta} \int_{\Omega} |u_j|^{p(x)-2} u_j u_j dx + \frac{\lambda}{\theta} \left( \int_{\partial \Omega} F(x, u_j) \right)^{r} \int_{\Omega} f(x, u_j) u_j + \frac{\gamma}{\theta} \int_{\partial \Omega} |u_j|^{q(x)-2} u_j u_j dS \]
\[- \frac{b}{\theta} \left( \int_{\Omega} \frac{1}{p(x)} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx \right)^{q} \left( \int_{\Omega} (|\nabla u_j|^{p(x)-2} \nabla u_j \nabla u_j + |u_j|^{p(x)-2} u_j u_j) dx \right) \]
Thus,
\[C + \|u_j\| \geq \left( \frac{a}{p^+} - \frac{a}{\theta} \right) \left( \int_{\Omega} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx \right) \]
\[+ \left( \frac{b}{(\eta + 1)(p^+)^{\eta+1}} - \frac{b}{\theta(p^-)^{\eta}} \right) \left( \int_{\Omega} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx \right)^{\eta+1} \]
\[+ \left( \frac{\lambda A_1^{r+1}}{\theta (\beta^+) r} - \frac{\lambda A_2^{r+1}}{(r+1)(\beta^-)^{r+1}} \right) \left[ \int_{\Omega} |u_j|^{\beta(x)} dx \right]^{\gamma+1} \]
\[\left( \frac{\gamma}{\theta} - \frac{\gamma}{q} \right) \int_{\partial \Omega} |u_j|^{q(x)} dS. \]
Now suppose that \(u_j\) is unbounded in \(W^{1,p(x)}(\Omega)\). Thus, passing to a subsequence if necessary, we get \(\|u_j\| > 1\) and we obtain
\[C + \|u_j\| \geq \left( \frac{a}{p^+} - \frac{a}{\theta} \right) \|u\|^{p^-} \left( \frac{b}{(\eta + 1)(p^+)^{\eta+1}} - \frac{b}{\theta(p^-)^{\eta}} \right) \|u\|^{(\eta+1)p^-} \]
\[+ \lambda \left( \frac{A_1^{r+1}}{\theta} \frac{1}{(\beta^+) r} - \frac{A_2^{r+1}}{r+1} \frac{1}{(\beta^-)^{r+1}} \right) \|u\|^{\beta^-(r+1)}, \]
which is a contradiction because \(\beta^+(r+1) > p^- > 1\). Hence \(u_j\) is bounded in \(W^{1,p(x)}(\Omega)\).

**Lemma 3.2** Let \((u_j) \subset W^{1,p(x)}(\Omega)\) be a Palais-Smale sequence, with energy level \(c\) and
\[t_0 = \lim_{j \to +\infty} \left( \int_{\Omega} \frac{1}{p(x)} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx \right). \]
If
\[c < \left( \frac{1}{\theta} - \frac{1}{q} \right) \inf_{x \in I} \left( \gamma^{1-1/p(x)} \frac{1}{\alpha^{1/p(x)}} T_{x_1} \right)^{\gamma(p(x)+1)(p(x)-1)}, \]
where \(\alpha = t_1\) with \(0 < t_1 < bt_0^0\), then index set \(I = \emptyset\) and \(u_j \to u\) strongly in \(L^{p(x)}(\partial \Omega)\).
Proof. Assume that $u_j \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. By Proposition 2.7 and Lemma 3.1, we have

$$|u_j|^{q(x)}dS \rightharpoonup \nu = |u|^{q(x)}dS + \sum_{i \in I} \nu_i \delta_{x_i}, \quad \text{weakly} - \ast \text{ in the sense of measures.}$$

$$|\nabla u_j|^{p(x)}dx \rightharpoonup \mu \geq |\nabla u|^{p(x)}dx + \sum_{i \in I} \mu_i \delta_{x_i}, \quad \text{weakly} - \ast \text{ in the sense of measures.}$$

$$T_i, \nu_i|^{1/\lambda_i} \leq \mu_i|^{1/\lambda_i}, \quad \forall i \in I.$$

If $I = \emptyset$ then $u_j \rightarrow u$ strongly in $L^{q(x)}(\partial \Omega)$. Suppose that $I \neq \emptyset$. Let $x_i$ be a singular point of the measures $\mu$ and $\nu$. We consider $\phi \in C_0^\infty(\mathbb{R}^N)$, such that $0 \leq \phi(x) \leq 1$, $\phi(0) = 0$ and support in the unit ball of $\mathbb{R}^N$. Consider the functions $\phi_{i,\varepsilon}(x) = \phi \left( \frac{x - x_i}{\varepsilon} \right)$ for all $x \in \mathbb{R}^N$ and $\varepsilon > 0$.

As $J'_{\lambda,\gamma}(u_j) \rightarrow 0$ in $\left(W^{1,p(x)}(\Omega)\right)'$ we obtain that

$$\lim J'_{\lambda,\gamma}(u_j)(\phi_{i,\varepsilon}u_j) = 0.$$
We may show that,
\[
\lim_{\varepsilon \to 0} (a + bt_0^\varepsilon) \int_\Omega (|\nabla u|^{p(x)} - 2 \nabla u \nabla \phi_{i,\varepsilon} u) \, dx \to 0, \text{ see Shang-Wang [18].}
\]
On the other hand,
\[
\lim_{\varepsilon \to 0} (a + bt_0^\varepsilon) \int_\Omega \phi_{i,\varepsilon} d\mu = (a + bt_0^\varepsilon) \mu(0), \quad \gamma \lim_{\varepsilon \to 0} \int_\Omega \phi_{i,\varepsilon} d\nu = \gamma \nu(0)
\]
and
\[
\lambda \left[ \int_\Omega F(x, u) \, dx \right] \geq \int_\Omega f(x, u)(\phi_{i,\varepsilon} u) \, dx \to 0, \quad (a + bt_0) \int_\Omega |u|^{p(x)} \phi_{i,\varepsilon} \, dx \to 0 \text{ as } \varepsilon \to 0.
\]

Then,
\[
(a + bt_0^\varepsilon) \mu(0) = \gamma \nu(0) \text{ implies that } \gamma^{-1} \pi \nu_i \leq \nu_i. \quad \text{By } \mathbf{T}_{x_i} \nu_i^{1/p(x_i)} \leq \nu_i^{1/p(x_i)} \text{ we obtain } \gamma^{-1} \pi \mathbf{T}_{x_i}^{p(x_i)} \nu_i^{p(x_i)/p(x_i)} \leq \gamma^{-1} \pi \nu_i \leq \nu_i. \quad \text{Thus } \gamma^{-1} \pi \mathbf{T}_{x_i}^{p(x_i)} \leq \nu_i^{(p(x_i) - p(x_i))/p(x_i)}/p(x_i) = \nu_i^{(p(x_i) - p(x_i))/p(x_i)} \text{ and } \gamma^{-1} \pi \mathbf{T}_{x_i} \leq \nu_i^{(p(x_i) - p(x_i))/p(x_i)}/p(x_i) \nu_i^{(p(x_i) - p(x_i))/p(x_i)}. \quad \text{Therefore}
\]
\[
\nu_i \geq \left( \gamma^{-1} \pi \mathbf{T}_{x_i}^{p(x_i)} \right)^{\frac{p(x_i)}{p(x_i) - p(x_i)}/p(x_i)}.
\]

On the other hand, using \( \theta \) satisfying (3.10)
\[
c = \lim J_{\lambda, \gamma}(u_j) = \lim \left( J_{\lambda, \gamma}(u_j) - \frac{1}{\theta} J'_{\lambda, \gamma}(u_j) u_j \right).
\]

Thus,
\[
c \geq \lim \left( \frac{\gamma}{\theta} - \frac{\gamma}{q^-} \right) \int_{\partial \Omega} |u_j|^q \, dS
\]
\[
c \geq \left( \frac{\gamma}{\theta} - \frac{\gamma}{q^-} \right) \left( \int_{\partial \Omega} |u_j|^q \, dS + \int_{\partial \Omega} \sum_{i \in I} \nu_i \delta_{x_i} \, d\nu \right) \geq \left( \frac{\gamma}{\theta} - \frac{\gamma}{q^-} \right) \nu_i
\]
\[
c \geq \left( \frac{\gamma}{\theta} - \frac{\gamma}{q^-} \right) \inf_{i \in I} \left( \gamma^{-1} \pi \mathbf{T}_{x_i}^{p(x_i)} \right)^{\frac{p(x_i)}{p(x_i) - p(x_i)}}.
\]

Therefore, the index set \( I \) is empty if, \( c < \left( \frac{1}{\theta} - \frac{1}{q^-} \right) \inf_{i \in I} \left( \gamma^{-1} \pi \mathbf{T}_{x_i}^{p(x_i)} \right)^{\frac{p(x_i)}{p(x_i) - p(x_i)}} \).

**Lemma 3.3** Let \( (u_j) \subset W^{1,p(x)}(\Omega) \) be a Palais-Smale sequence with energy level \( c \).
If \( c < \left( \frac{1}{\theta} - \frac{1}{q^-} \right) \inf_{i \in I} \left( \gamma^{-1} \pi \mathbf{T}_{x_i}^{p(x_i)} \right)^{\frac{p(x_i)}{p(x_i) - p(x_i)}} \), there exist \( u \in W^{1,p(x)}(\Omega) \) and a subsequence, still denoted by \( (u_j) \), such that \( u_j \to u \) in \( W^{1,p(x)}(\Omega) \).

**Proof.** From
\[
J'_{\lambda, \gamma}(u_j) \to 0,
\]
we have
\[ J_{\lambda, \gamma}(u_j) \phi u_j = \left( a + b \left( \int_{\Omega} \frac{1}{p(x)} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) \, dx \right)^{\eta} \right) \]

\[ \times \int_{\Omega} \left( |\nabla u_j|^{p(x)-2} \nabla u_j \nabla (u_j - u) + |u_j|^{p(x)-2} u_j (u_j - u) \right) \, dx \]

\[ - \gamma \int_{\partial \Omega} |u_j|^{q(x)-2} u_j (u_j - u) \, dS \]

\[ - \lambda \left[ \int_{\Omega} F(x, u_j) \, dx \right]^r \int_{\Omega} f(x, u_j) (u_j - u) \, dx \to 0, \]

Note that there exists nonnegative constants \( c_1, c_2, c_3 \) and \( c_4 \) such that

\[ c_1 \leq a + b \left( \int_{\Omega} \frac{1}{p(x)} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) \, dx \right)^{\eta} \leq c_2 \]

and

\[ c_3 \leq \left[ \int_{\Omega} F(x, u_j) \, dx \right]^r \leq c_4. \]

But using Hölder inequality, we have

\[ \left| \int_{\Omega} |u_j|^{p(x)-2} u_j (u_j - u) \, dx \right| \leq \int_{\Omega} |u_j|^{p(x)-1} |u_j - u| \, dx \leq C_1 |u|_{p(x)/p(x)-1} |u_j - u|_{p(x)}, \]

and

\[ \left| \int_{\Omega} f(x, u_j) (u_j - u) \, dx \right| \leq A_2 \int_{\Omega} |u_j|^{\beta(x)-1} |u_j - u| \, dx \leq C_2 |u|_{\beta(x)/\beta(x)-1} |u_j - u|_{\beta(x)}, \]

where \( C_1 \) and \( C_2 \) are positive constants. Thus

\[ \left| \int_{\Omega} |u_j|^{p(x)-2} u_j (u_j - u) \, dx \right| \to 0 \]

and

\[ \left| \int_{\Omega} f(x, u_j) (u_j - u) \, dx \right| \to 0. \]

By Lemma 3.2, \( u_j \to u \) in \( L^{q(x)}(\partial \Omega) \) and using Hölder inequality we obtain

\[ \left| \int_{\partial \Omega} |u_j|^{q(x)-2} u_j (u_j - u) \, dx \right| \to 0. \]

If we take

\[ L_{p(x)}(u_j)(u_j - u) = \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla (u_j - u) \, dx, \]

we obtain \( L_{p(x)}(u_j)(u_j - u) \to 0 \). We also have \( L_{p(x)}(u)(u_j - u) \to 0 \). So

\[ (L_{p(x)}(u_j) - L_{p(x)}(u), u_j - u) \to 0. \]

From Proposition 2.6 we have \( u_j \to u \) in \( W^{1,p(x)}(\Omega) \).
Lemma 3.4

(i) For all $\lambda > 0$, there are $\alpha > 0, \rho > 0$, such that $J_{\lambda,\gamma}(u) \geq \alpha, \|u\| = \rho$.

(ii) There exists an element $w_0 \in W^{1,p(x)}(\Omega)$ with $\|w_0\| > \rho$ and $J_{\lambda,\gamma}(w_0) < \alpha$.

Proof. (i) We have,

$$J_{\lambda,\gamma}(u) \geq a \left( \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \right)$$

$$+ \frac{b}{\eta + 1} \left( \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \right)^{\eta + 1}$$

$$- \frac{\lambda}{r + 1} \left[ \int_{\Omega} \frac{A_2}{\beta} u^{\beta(x)} \, dx \right]^{r + 1} - \frac{\gamma}{q} \int_{\partial \Omega} |u|^q(x) \, dS.$$ 

So,

$$J_{\lambda,\gamma}(u) \geq a \left( \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \right)$$

$$+ \frac{b}{(\eta + 1)(p^+)^{\eta + 1}} \left( \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \right)^{\eta + 1}$$

$$- \frac{\lambda}{r + 1} \left( \frac{A_2}{\beta} \right)^{r + 1} M_1^{1\beta} \|u\|^{\beta(r + 1)} - \frac{\gamma}{q} M_2^q \|u\|^q.$$

If $\|u\| < 1$ is small enough, from Proposition 2.3 we obtain

$$\int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) \geq \|u\|^{p^+}$$

and from the immersions $W^{1,p(x)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$ and $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial \Omega)$ (continuous), we get

$$\int_{\Omega} |u|^\beta(x) \, dx \leq M_1^{1\beta} \|u\|^\beta$$

and

$$\int_{\partial \Omega} |u|^q(x) \, dS \leq M_2^q \|u\|^q.$$

Hence

$$J_{\lambda,\gamma}(u) \geq a \left( \frac{p^+}{(\eta + 1)(p^+)^{\eta + 1}} \|u\|^{p^+(\eta + 1)} \right)$$

$$- \frac{\lambda}{r + 1} \left( \frac{A_2}{\beta} \right)^{r + 1} M_1^{1\beta} \|u\|^{\beta(r + 1)} - \frac{\gamma}{q} M_2^q \|u\|^q.$$

Using the assumption $(\eta + 1)p^+ < \beta^-(r + 1) < q^-$,

$$J_{\lambda,\gamma}(u) \geq a \left( \frac{p^+}{(\eta + 1)(p^+)^{\eta + 1}} \|u\|^{(\eta + 1)p^+} \right)$$

$$- \frac{\lambda}{r + 1} \left( \frac{A_2}{\beta} \right)^{r + 1} M_1\beta^{(r + 1)} \|u\|^\beta - \frac{\gamma}{q} M_2^q \|u\|^q - \frac{\lambda}{r + 1} \left( \frac{A_2}{\beta} \right)^{r + 1} M_1^{1\beta} \|u\|^{\beta(r + 1)},$$

can be written as

$$J_{\lambda,\gamma}(u) \geq a \left( \frac{p^+ + (\eta + 1)(p^+)^{\eta + 1}}{(\eta + 1)(p^+)^{\eta + 1}} \|u\|^{p^+(\eta + 1)} \right)$$

$$- \left( \frac{\lambda}{r + 1} \left( \frac{A_2}{\beta} \right)^{r + 1} M_1^{1\beta} + \frac{\gamma}{\beta^-(r + 1)} M_2^q \right) \|u\|^{\beta(r + 1)}.$$ 

Taking $\rho = \|u\|$,  

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\[ J_{\lambda, \gamma}(u) \geq \rho^{(\eta+1)p^+} \left[ \left( \frac{a}{p^+} + \frac{b}{(\eta + 1)(p^+)^{\eta+1}} \right) \lambda \left( 1 + \frac{A_2}{\beta^+} \right)^{r+1} M_1^{\beta^-(r+1)} + \frac{\gamma}{\beta^-(r+1)} M_2^{\beta^-(r+1)-(\eta+1)p^+} \right], \]

and the result follows.

\((ii)\) Take \(0 < w \in W^{1,p(x)}(\Omega).\) For \(t > 1,\)
\[ J_{\lambda, \gamma}(tw) \leq \frac{a}{p^+} t^{p^+} \left( \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right) \]
\[ + \frac{b t^{p^+} (\eta + 1)(p^+)^{\eta+1}}{p^+ (\eta + 1)(p^+)^{\eta+1}} \left( \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right)^{\eta+1} \]
\[ + \frac{\lambda}{r+1} \left( 1 + \frac{A_2}{\beta^+} \right)^{r} t^{\beta^-(r+1)} \left( \int_{\Omega} |w|^{\beta(x)} \, dx \right)^{r+1} - \frac{\gamma}{q^+} t^{q^+} \int_{\partial \Omega} |w|^{q(x)} \, dS. \]

Then we have
\[ \lim_{t \to \infty} J_{\lambda, \gamma}(tw) = -\infty, \]
and the proof is over.

\[ \square \]

By Lemma 3.4, we may use the Mountain Pass Theorem [19], which guarantees the existence of a sequence \((u_j, y_j) \subset W^{1,p(x)}(\Omega)\) such that
\[ J_{\lambda, \gamma}(u_j) \to c \quad \text{and} \quad J_{\lambda, \gamma}'(u_j) \to 0 \quad \text{in} \quad (W^{1,p(x)}(\Omega))', \]
where a candidate for critical value is
\[ c = \inf_{h \in \mathcal{C}} \sup_{t \in [0,1]} J_{\lambda, \gamma}(h(t)) \]
and \(\mathcal{C} = \{ h : [0, 1] \to W^{1,p(x)}(\Omega) : h \text{ continuous and } h(0) = 0, h(1) = w_0 \}.\)

For \(0 < t < 1\) and fixing \(w \in W^{1,p(x)}(\Omega),\)
\[ J_{\lambda, \gamma}(tw) \leq \frac{a}{p^+} t^{p^+} \left( \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right) \]
\[ + \frac{b t^{p^+} (\eta + 1)(p^+)^{\eta+1}}{p^+ (\eta + 1)(p^+)^{\eta+1}} \left( \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right)^{\eta+1} \]
\[ - \frac{\lambda}{r+1} \left( 1 + \frac{A_2}{\beta^+} \right)^{r} t^{\beta^-(r+1)} \left( \int_{\Omega} |w|^{\beta(x)} \, dx \right)^{r+1} - \frac{\gamma}{q^+} t^{q^+} \int_{\partial \Omega} |w|^{q(x)} \, dS. \]

\[ J_{\lambda, \gamma}(tw) \leq \frac{a}{p^+} t^{p^+} \left( \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right) \]
\[ + \frac{b t^{p^+} (\eta + 1)(p^+)^{\eta+1}}{p^+ (\eta + 1)(p^+)^{\eta+1}} \left( \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right)^{\eta+1} \]
\[ - \frac{\lambda}{r+1} \left( 1 + \frac{A_2}{\beta^+} \right)^{r} t^{\beta^-(r+1)} \left( \int_{\Omega} |w|^{\beta(x)} \, dx \right)^{r+1}. \]
Setting \( g(t) = \left( \frac{a\hat{a}}{p^{-}} + \frac{b\hat{a}^{\eta+1}}{(\eta + 1)(p^{-})^{\eta+1}} \right) t^{p^{-}} - \frac{\lambda}{r+1} \left( \frac{A_{1}}{\beta^{+}} \right) \hat{b}^{\beta-(r+1)}, \) where \( \hat{a} = \int_{\Omega} (|\nabla w|^{p(x)} + |w|^{p(x)}) dx \) and \( \hat{b} = \int_{\Omega} |w|^{\beta(x)} dx, \) we obtain the following inequality:

\[
\frac{1}{\hat{a}^{\eta+1}} \frac{A_{1}}{(\eta + 1)(p^{-})^{\eta+1}} \hat{b}^{\beta-(r+1)} \leq g(t).
\]

Note that \( g(t) \) has a critical point of maximum

\[
t_\lambda = \left( \frac{\beta^{+}((\eta + 1)(p^{-})^{\eta})a\hat{a} + b(\hat{a})^{\eta+1}}{\lambda A_{1}(\eta + 1)(p^{-})^{\eta}} \right) \frac{1}{\beta^{+}(r+1)-p^{-}},
\]

and that \( t_\lambda \to 0 \) when \( \lambda \to \infty. \) By the continuity of \( J_{\lambda,\gamma} \)

\[
\lim_{\lambda \to \infty} \left( \sup_{t \geq 0} J_{\lambda,\gamma}(tw) \right) = 0.
\]

Then exists \( \lambda_0 \) such that \( \forall \lambda \geq \lambda_0 \)

\[
\sup_{t \geq 0} J_{\lambda,\gamma}(tw) < \left( \frac{1}{\theta} - \frac{\gamma}{q} \right) \inf_{\xi \in I} \left( \gamma^{1-1/p(x_{1})}\xi^{1/p(x_{1})} \right)^{\frac{\mu(x_{1})p_{\xi}(x_{1})}{p_{\xi}(x_{1})-p(x_{1})}}
\]

This completes the proof of part (i).

\[\n\]

**Proof.** (ii) Like the item (i).

**Proof.** (iii) The proof follows from several lemmas.

**Lemma 3.5** Let \( (u_j) \subset W^{1,p(x)}(\Omega) \) be a Palais-Smale sequence for \( J_{\lambda,\gamma} \) then \( (u_j) \) is bounded.

**Proof.** Let \( (u_j) \) be a Palais-Smale sequence with energy level \( c, \) i.e., \( J_{\lambda,\gamma}(u_j) \to c \) and \( J'_{\lambda,\gamma}(u_j) \to 0. \) Then taking \( \theta \) such that

\[
\frac{m_{1}p^{+}}{m_{0}} < \theta < \left( \frac{A_{1}}{A_{2}} \right)^{\kappa+1} \frac{(\beta^{-})^{\kappa+1}(\kappa + 1)}{(\beta^{+})^{\kappa}}, \quad \tag{3.11}
\]

we have

\[
C + \|u_j\| \geq \left( J_{\lambda,\gamma}(u_j) - \frac{1}{\theta} J'_{\lambda,\gamma}(u_j)u_j \right).
\]

So

\[
C + \|u_j\| \geq \left( \frac{m_{0}}{p^{+}} - \frac{m_{1}}{\theta} \right) \left( \int_{\Omega} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx \right) + \left( \frac{\lambda}{\theta} - \frac{\lambda}{q^{+}} \right) \int_{\Omega} |u_j|^{q(x)} dx + \left( \frac{\gamma A_{1}^{\kappa+1}}{\theta(\beta^{+})^{\kappa}} - \frac{\gamma A_{2}^{\kappa+1}}{(\kappa + 1)(\beta^{-})^{\kappa+1}} \right) \left[ \int_{\partial \Omega} |u|^{\beta(x)} dS \right]^{\kappa+1}.
\]

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Now suppose that \((u_j)\) is unbounded in \(W^{1,p(x)}(\Omega)\). Thus, passing to a subsequence if necessary, we get \(\|u_j\| > 1\) and we obtain
\[
C + \|u_j\| \geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_j\|^{p^-},
\]
which is a contradiction because \(p^- > 1\). Hence \((u_j)\) is bounded in \(W^{1,p(x)}(\Omega)\).

**Lemma 3.6** Let \((u_j) \subset W^{1,p(x)}(\Omega)\) be a Palais-Smale sequence, with energy level \(c\). If 
\[
c < \left(\frac{1}{\theta} - \frac{1}{q_A}\right)\lambda(m_0S)^N\]
where \(m_0 = \min\{\lambda^{-1}m_0^{1/p^+}, (\lambda^{-1}m_0)^{1/p^-}\}\), then index set 
\(I = \emptyset\) and \(u_j \rightharpoonup u\) strongly in \(L^{q(x)}(\Omega)\).

**Proof.** Assume that \(u_j \rightharpoonup u\) weakly in \(W^{1,p(x)}(\Omega)\). By Proposition 2.8 and Lemma 3.5, we have
\[
|u_j|^{q(x)} \rightharpoonup \nu = |u|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, \quad \nu_i > 0
\]
and
\[
|\nabla u_j|^{p(x)} \rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i}, \quad \mu_i > 0
\]
with
\[
S \nu_i^{1/p'(x_i)} \leq \mu_i^{1/p'(x_i)}, \quad \forall i \in I.
\]

If \(I = \emptyset\) then \(u_j \rightharpoonup u\) strongly in \(L^{q(x)}(\Omega)\). Suppose that \(I \neq \emptyset\). Let \(x_i\) be a singular point of the measures \(\mu\) and \(\nu\). We consider \(\phi \in C^\infty_0(\mathbb{R}^N)\), such that \(0 \leq \phi(x) \leq 1\), \(\phi(0) = 0\) and and support in the unit ball of \(\mathbb{R}^N\). Consider the functions \(\phi_{i,\varepsilon}(x) = \phi\left(\frac{x - x_i}{\varepsilon}\right)\) for all \(x \in \mathbb{R}^N\) and \(\varepsilon > 0\).

As \(J'_{\lambda,\gamma}(u_j) \to 0\) in \(W^{1,p(x)}(\Omega)'\) we obtain that
\[
\lim_j J'_{\lambda,\gamma}(u_j)(\phi_{i,\varepsilon}u_j) = 0.
\]

When \(j \to \infty\) we get
\[
0 = \lim_j \left(M\left(\int_{\Omega} \frac{1}{p(x)}(|\nabla u_j|^{p(x)} + |u_j|^{p(x)})dx\right)\int_{\Omega} |\nabla u_j|^{p(x)}|\nabla \phi_{i,\varepsilon}u_j|dx + \gamma \int_{\partial \Omega} G(x, u_j)dS\right) + \int_{\partial \Omega} g(x, u_j)(\phi_{i,\varepsilon}u_j)dS \to 0.
\]

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\[ M(t_0) \int_\Omega \phi_{i,\varepsilon} d\mu - \lambda \int_\Omega \phi_{i,\varepsilon} d\nu - \gamma \int_{\partial \Omega} G(x, u) dS \int_{\partial \Omega} g(x, u) (\phi_{i,\varepsilon} u) dS, \]

where \( t_0 = \lim_{j \to \infty} \left( \int_\Omega \frac{1}{p(x)} (|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx \right). \)

We may show that,

\[ \lim_{\varepsilon \to 0} M(t_0) \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla \phi_{i,\varepsilon} u) dx \to 0, \text{ see Shang-Wang } [18]. \]

On the other hand,

\[ \lim_{\varepsilon \to 0} M(t_0) \int_\Omega \phi_{i,\varepsilon} d\mu = M(t_0) \mu \phi(0), \quad \lambda \lim_{\varepsilon \to 0} \int_\Omega \phi_{i,\varepsilon} d\nu = \nu \phi(0) \]

and

\[ \gamma \left( \int_{\partial \Omega} G(x, u) dS \right) \int_{\partial \Omega} g(x, u) (\phi_{i,\varepsilon} u) dS \to 0, \quad M(t_0) \int_\Omega |u|^{p(x)} \phi_{i,\varepsilon} dx \to 0 \text{ quando } \varepsilon \to 0. \]

Then,

\[ M(t_0) \mu_i \phi(0) = \lambda \nu_i \phi(0) \text{ implies that } \lambda^{-1} m_0 \mu_i \leq \nu_i. \]

By \( S \nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)} \) we get

\[ (\overline{m_0} S)^N \leq \nu_i, \]

where \( \overline{m_0} = \min \left\{ (\lambda^{-1} m_0)^{1/p^*}, (\lambda^{-1} m_0)^{1/p^-} \right\}. \)

On the other hand, using \( \theta \) satisfying (3.11) we have

\[ c = \lim \left( J_\lambda(u_j) - \frac{1}{\theta} J'_{\lambda,\gamma}(u_j) u_j \right). \]

Note that,

\[ c \geq \lim \int_\Omega \left( \frac{\lambda}{\theta} - \frac{\lambda}{q(x)} \right) |u_j|^{p(x)} dx. \]

Considering \( A_\delta = \bigcup_{x \in A} (B_\delta(x) \cap \Omega) = \{ x \in \Omega : \text{dist}(x, A) < \delta \}, \)

\[ c \geq \left( \frac{\lambda}{\theta} - \frac{\lambda}{q_{A_\delta}} \right) \left( \int_{A_\delta} |u|^{q(x)} dx + \sum_{i \in I} \nu_i \right) \geq \left( \frac{\lambda}{\theta} - \frac{\lambda}{q_{A_\delta}} \right) \nu_i. \]

Therefore,

\[ c \geq \left( \frac{\lambda}{\theta} - \frac{\lambda}{q_A} \right) (\overline{m_0} S)^N. \]

Therefore, the index set I is empty if,

\[ c < \left( \frac{1}{\theta} - \frac{1}{q_A} \right) \lambda (\overline{m_0} S)^N. \]
Lemma 3.7 Let \((u_j) \subset W^{1,p(x)}(\Omega)\) be a Palais-Smale sequence with energy level \(c\). If 
\[c < \left(\frac{1}{\theta} - \frac{1}{q_A}\right) \lambda(m_0 S)^N,\]
there exist \(u \in W^{1,p(x)}(\Omega)\) and a subsequence, still denoted by \((u_j)\), such that \(u_j \to u\) in \(W^{1,p(x)}(\Omega)\).

**Proof.** From 
\[J'_{\lambda,\gamma}(u_j) \to 0,\]
we have
\[J'_{\lambda,\gamma}(u_j)(u_j - u) = M \left(\int_{\Omega} \frac{1}{p(x)}(|\nabla u_j|^{p(x)} + |u_j|^{p(x)}) dx\right) \times \int_{\Omega} (|\nabla u_j|^{p(x)-2}\nabla u_j \nabla (u_j - u) + |u_j|^{p(x)-2}u_j(u_j - u)) dx \]
\[-\lambda \int_{\Omega} |u_j|^{q(x)-2}u_j (u_j - u) dx \]
\[-\gamma \left[\int_{\partial \Omega} G(x, u_j) dS\right] \int_{\partial \Omega} g(x, u_j)(u_j - u) dS \to 0.\]

Note that there exists nonnegative constants \(c_1\) and \(c_2\) such that
\[c_1 \leq \left[\int_{\partial \Omega} G(x, u_j) dS\right]^r \leq c_2.\]

Using Hölder inequality, one has
\[\left|\int_{\Omega} |u_j|^{p(x)-2}u_j (u_j - u) dx\right| \leq \int_{\Omega} |u_j|^{p(x)-1} |u_j - u| dx \leq C_1 \|u\|_{p(x)/p(x)-1} |u_j - u|_{p(x)},\]
and
\[\int_{\partial \Omega} g(x, u_j)(u_j - u) dS \leq A_2 \int_{\partial \Omega} |u_j|^{\beta(x)-1} |u_j - u| dS \leq C_2 \|u\|_{\beta(x)-1}^{\beta(x)/\beta(x)-1} |u_j - u|_{\beta(x)},\]
where \(C_1\) and \(C_2\) are positive constants. Thus
\[\left|\int_{\Omega} |u_j|^{p(x)-2}u_j (u_j - u) dx\right| \to 0\]
and
\[\left|\int_{\partial \Omega} g(x, u_j)(u_j - u) dS\right| \to 0.\]

By Lemma 3.6, \(u_j \to u\) in \(L^q(x)(\Omega)\) and using Hölder inequality we obtain
\[\left|\int_{\Omega} |u_j|^{q(x)-2}u_j (u_j - u) dx\right| \to 0.\]

If we take
\[ L_{p(x)}(u_j)(u_j - u) = \int_{\Omega} |\nabla u_j|^p(x) - 2 \nabla u_j \nabla (u_j - u), \]
we obtain \( L_{p(x)}(u_j)(u_j - u) \to 0. \) We also have \( L_{p(x)}(u)(u_j - u) \to 0. \) So
\[
(L_{p(x)}(u_j) - L_{p(x)}(u))(u_j - u) \to 0.
\]
From Proposition 2.6 we have \( u_j \to u \) in \( W^{1,p(x)}(\Omega). \)

\[ \]

Lemma 3.8

(i) For all \( \lambda > 0, \) there are \( \alpha > 0, \rho > 0, \) such that \( J_{\lambda, \gamma}(u) \geq \alpha, \) \( \|u\| = \rho. \)

(ii) There exists an element \( w_0 \in W^{1,p(x)}(\Omega) \) with \( \|w_0\| > \rho \) and \( J_{\lambda, \gamma}(w_0) < \alpha. \)

Proof. (i) We have,
\[
J_{\lambda, \gamma}(u) \geq m_0 \left( \int_{\Omega} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx \right) - \frac{\gamma}{\kappa + 1} \left( \int_{\Omega} \frac{A_2}{\beta(x)} u^{\beta(x)}dS \right)^{\kappa + 1} - \frac{\lambda}{q^{-}} \int_{\Omega} |u|^{q(x)}dx.
\]
So,
\[
J_{\lambda, \gamma}(u) \geq m_0 \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \frac{\gamma}{\kappa + 1} \left( \frac{A_2}{\beta^{-}} \right)^{\kappa + 1} \left[ \int_{\Omega} u^{\beta(x)}dS \right]^{\kappa + 1} - \frac{\lambda}{q^{-}} \int_{\Omega} |u|^{q(x)}dx.
\]
If \( \|u\| < 1 \) is small enough, from Proposition 2.3 we obtain \( \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) \geq \|u\|^{p^{-}} \) and from the immersions \( W^{1,p(x)}(\Omega) \hookrightarrow L^{\beta^{-}}(\Omega) \) and \( W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \) (continuous), we get \( \int_{\Omega} |u|^{\beta(x)}dx \leq M_1^{\beta^{-}} \|u\|^{\beta^{-}} \) and \( \int_{\Omega} |u|^{q(x)}dx \leq M_2^{q^{-}} \|u\|^{q^{-}}. \)

Hence
\[
J_{\lambda, \gamma}(u) \geq m_0 \frac{p^{-}}{p^{+}} \|u\|^{p^{-}} - \frac{\gamma}{\kappa + 1} \left( \frac{A_2}{\beta^{-}} \right)^{\kappa + 1} M_1^{\beta^{-}(\kappa+1)} \|u\|^{\beta^{-}(\kappa+1)} - \frac{\lambda}{q^{-}} M_2^{q^{-}} \|u\|^{q^{-}}.
\]
Using the assumption \( p^{+} < \beta^{-}(\kappa + 1) < q^{-}, \)
\[
J_{\lambda, \gamma}(u) \geq m_0 \frac{p^{-}}{p^{+}} \|u\|^{p^{-}} - \frac{\gamma}{\kappa + 1} \left( \frac{A_2}{\beta^{-}} \right)^{\kappa + 1} M_1^{\beta^{-}(\kappa+1)} \|u\|^{\beta^{-}(\kappa+1)} - \frac{\lambda}{\beta^{-}(\kappa + 1)} M_2^{q^{-}} \|u\|^{q^{-}}.
\]
can be written as
\[
J_{\lambda, \gamma}(u) \geq m_0 \frac{p^{-}}{p^{+}} \|u\|^{p^{-}} - \left( \frac{\gamma}{\kappa + 1} \left( \frac{A_2}{\beta^{-}} \right)^{\kappa + 1} M_1^{\beta^{-}(\kappa+1)} + \frac{\lambda}{\beta^{-}(\kappa + 1)} M_2^{q^{-}} \right) \|u\|^{\beta^{-}(\kappa+1)}.
\]
Taking \( \rho = \|u\|, \)
\[ J_{\lambda,\gamma}(u) \geq \rho^{p^+} \left[ \left( \frac{m_0}{p^+} \right) - \left( \frac{\gamma}{\kappa + 1} \left( \frac{A_2}{\beta^-} \right)^{\kappa + 1} M_1^{\beta^-} + \frac{\lambda}{\beta^-} M_2^{\beta^-} \right) \rho^{\beta^- (\kappa + 1) - p^+} \right], \]

and the result follows.

(ii) Take \( 0 < w \in W^{1,p(x)}(\Omega) \). For \( t > 1 \),
\[
J_{\lambda,\gamma}(tw) \leq \frac{m_1}{p^+} t^{p^+} \left( \int_\Omega (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right) - \frac{\lambda}{q^+} t^{q^-} \int_\Omega |w|^{q(x)} \, dx \\
+ \frac{\gamma}{\kappa + 1} \left( \frac{A_2}{\beta^-} \right)^{\kappa + 1} t^{\beta^- (\kappa + 1)} \left( \int_\partial \Omega |w|^{\beta(x)} \, dS \right)^{\kappa + 1}.
\]

Then we have
\[
\lim_{t \to \infty} J_{\lambda,\gamma}(tw) = -\infty,
\]

and the proof is over.

By Lemma 3.4, we may use the Mountain Pass Theorem [19], which guarantees the existence of a sequence \( (u_j) \subset W^{1,p(x)}(\Omega) \) such that
\[
J_{\lambda,\gamma}(u_j) \to c \quad \text{and} \quad J_{\lambda,\gamma}'(u_j) \to 0 \quad \text{in} \quad (W^{1,p(x)}(\Omega))',
\]

where a candidate for critical value is
\[ c = \inf_{h \in C} \sup_{t \in [0,1]} J_{\lambda,\gamma}(h(t)) \]

and \( C = \{h : [0,1] \to W^{1,p(x)}(\Omega) : h \text{ continuous and } h(0) = 0, h(1) = w_0\} \).

For \( 0 < t < 1 \) and fixing \( w \in W^{1,p(x)}(\Omega) \),
\[
J_{\lambda,\gamma}(tw) \leq \frac{m_1}{p^+} t^{p^+} \left( \int_\Omega (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \right) - \frac{\lambda}{q^+} t^{q^-} \int_\Omega |w|^{q(x)} \, dx \\
- \frac{\gamma}{\kappa + 1} \left( \frac{A_1}{\beta^+} \right)^{\kappa + 1} t^{\beta^+ (\kappa + 1)} \left( \int_\partial \Omega |w|^{\beta(x)} \, dS \right)^{\kappa + 1}.
\]

Setting \( g(t) = \frac{m_1 \tilde{a}}{p^+} t^{p^+} - \frac{\gamma}{\kappa + 1} \left( \frac{A_1}{\beta^+} \right)^{\kappa + 1} \tilde{b} t^{\beta^+ (\kappa + 1)} \),

where \( \tilde{a} = \int_\Omega (|\nabla w|^{p(x)} + |w|^{p(x)}) \, dx \) and \( \tilde{b} = \int_\Omega |w|^{\beta(x)} \, dx \), we obtain \( \sup J_{\lambda,\gamma}(tw) \leq g(t) \).

Note that \( g(t) \) has a critical point of maximum
\[
t_\gamma = \left[ \frac{m_1 (\beta^+)^\kappa \tilde{a}}{\gamma A_1^2 \tilde{b}} \right]^{\frac{1}{\beta^+ (\kappa + 1) - p^+}},
\]

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and that $t_\gamma \to 0$ when $\gamma \to \infty$. By the continuity of $J_{\lambda,\gamma}$

$$
\lim_{\gamma \to \infty} \left( \sup_{t \geq 0} J_{\lambda,\gamma}(tw) \right) = 0.
$$

Then exists $\gamma_1$ such that $\forall \gamma \geq \gamma_1$

$$
\sup_{t \geq 0} J_{\lambda,\gamma}(tw) < \left( \frac{1}{\theta} - \frac{1}{q_A} \right) \lambda (m_0 S)^N.
$$

This completes the proof of part (iii).

\begin{flushright}
$\blacksquare$
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**Proof.** (iv) Similar to the proof of the item (iii).

**Remark 3.1** In a forthcoming article we will present results concerning multiplicity of solutions for the problem (1.1).
References


